

How classical is quantum mechanics?

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Abstract

Scrutinizing Schrödinger's derivation we find that the notion of the alleged breach of the laws of classical mechanics cannot be upheld. For *closed* systems, only a shift of the point of view is necessary. Away from the established picture of a point mass moving in phase space toward a pure energy representation of this motion: Applying a variational procedure to a special form of the Hamilton-Jacobi equation Schrödinger aimed at identifying extreme values of the mechanical *total* energy in closed systems. Known as belonging to stationary states the energy eigenvalues to his *time-independent* equation are these extreme values. They are shown to represent extraordinary situations of dynamic balance. Their mechanical features require that they be independent of time and their number countable. Lacking any accelerations, they cannot emit electromagnetic radiation.

For the description of the general dynamic behavior in *closed* systems in terms of an energy representation, contrary to Schrödinger's original intention, the expansion must be applied to the eigensolutions of his *time-dependent* equation. Conserving the *total* energy at any time, the resultant time-dependent state vectors entail time-dependent *general* expectation values for the constituent energies, which periodically exchange a certain amount of energy in accordance with the classical Virial theorem for the respective potential. The reason why they do so is that the total time derivative of these *general* expectation values behaves analogously to the classical equation of motion of an arbitrary function where Poisson's bracket is equivalently replaced with the respective commutator. This survey mentions several, mostly wellknown examples details of which may be found elsewhere.

1 Introduction

From its very beginning, quantum mechanics has been plagued by epistemological difficulties: Be it Bohr's wondering about "peculiar" stationary states [1, 2], which not only he considered to exclusively represent physical reality in spite of allegedly being in contradiction to classical electrodynamics, be it what has become known as the measurement problem [3], von Neumann's insistence on the pretended fact that, as a consequence, state vectors have to "collapse" upon measurement or be it the hypothesis of instantaneous "quantum jumps" [1, 2, 4–7]. All these ideas have led Bohr [8, 9] and many other early proponents of quantum mechanics [10, 13–22] to the commonly shared conviction that it is not possible in the atomic realm to describe continuous space-time dynamics. In their view, this discontinuity has to be assumed as a consequence of the fact that since Planck's discovery [23, 24] the energy is deemed only capable of discontinuous, so-to-say, "piece-wise" changes [2, 5, 8, 9, 16, 18, 25–27]. Hence, spurred by the idea that everything in the atomic realm must be different and in order to somehow make ends meet Heisenberg proposed the necessity of a search for new kinematic and mechanical relations [10, 14–17, 28]. The well-known results of this search are his "uncertainty relations" [5, 19], which were and still are being considered to confirm this "discontinuum" theory. Especially the "uncertainty relations" are deemed to justify the assumption of the alleged indeterminacy, i.e., the absence of causality as a result of the inherent fundamental impossibility to describe the dynamics of *individual* particles [13, 18, 20, 26, 29, 30]. Therefore, according to Heisenberg [28], one should only concentrate on observables, i.e., properties, which he considered measurable, like energies, frequencies, and transition probabilities and not care about descriptiveness or even try to explain the dynamics in *individual* particles.

What is puzzling about this alleged deep division between classical and quantum mechanics is the fact that Schrödinger's equations [31–35], actually *the* key foundations of quantum mechanics, are deeply rooted in classical mechanics. Especially his *time-dependent* equation [35], a linear partial differential equation in space *and* time, is synonymous for *continuous* action, i.e., for determinacy and causality, of the properties *represented* by it. The widely accepted Copenhagen interpretation, however, only allows these features to apply to his so-called wave function in terms of a conservation of probability density. Nevertheless, Schrödinger denounced the ideas of discontinuous behavior in the atomic world. Legendary is his dispute with Bohr

during his visit in Copenhagen about “quantum jumps” [37]. Despite all the general appreciation of his findings, Schrödinger’s stand was criticized by many of his contemporaries and his achievements were even denounced as quasi-classical [20, 21] or distracting [5] in spite of acknowledging [15, 20, 21] that the seemingly diametrically opposite approaches by him and Heisenberg had independently been proven to be equivalent [3, 33, 38].

Even nowadays, 94 years after Schrödinger’s discovery, there is widespread confusion about how classical and quantum mechanics correlate despite the fact that objective, in some sense even classical pictures are being used in order to understand phenomena that are not accessible to human senses.

However, with all the mathematical tools that have been developed since, we shall show, based on Schrödinger’s equations, that it is possible to describe continuous behavior in a very classical way: While the eigensolutions to Schrödinger’s *time-independent* equation exclusively isolate the *total* energies of extraordinary situations of dynamic balance the expansion applied to the eigensolutions of Schrödinger’s *time-dependent* equation embraces all kinds of dynamic behavior from an energy perspective by primarily allowing to describe multiple periodic energy exchanges as part of the general nonstationary dynamics in *closed* systems. This will be our focus here.

While here only a gross account can be given, many familiar examples and detailed calculations may be found in [39, 40]. At this point, we want to draw attention especially to Part II in [39] where we demonstrate that the *complete* time evolution of a *spontaneous* transition between two isolated energy levels can be described as being driven by unbalanced internal dynamics. Without having to resort to QED, this description is capable of naturally explaining all features found experimentally.

2 Schrödinger’s derivation

2.1 The classical starting point

For illustrating how in the early days of quantum mechanics the prevalent *zeitgeist* influenced the interpretation of its results and to prove our claim we shall take a very close look at Schrödinger’s derivation. However, before blaming anyone for anything we should get aware of the fact that until about 120 years ago the physical thinking in mechanics was dominated by the imagination of a mass point moving in phase space and, therefore, by equations

of motion. This may be the reason why Heisenberg, with his “uncertainty relations” in mind [5, 17–19], recommended that one should no longer try to find ways of how to calculate, for instance, basically unobservable electron orbits, or, as Dirac put it in following Heisenberg’s opinion [41], “science is concerned only with observable things” and “what cannot be investigated by experiment, should be regarded as outside the domain of science”. Instead, one should concentrate on establishing relations between truly observable properties as, according to Heisenberg’s conviction, only represented by the elements of diagonal matrices [5, 14, 41–43], i.e., the expectation values of *stationary* properties in the respective *stationary* states.

In order to understand how it all began we start with the well-known canonical equations of classical mechanics, i.e. Hamilton’s equations

$$\dot{q}_k = \frac{\partial H(p_k, q_k)}{\partial p_k}, \quad \dot{p}_k = -\frac{\partial H(p_k, q_k)}{\partial q_k} \quad (1)$$

where Hamilton’s function

$$H(p_k, q_k, t) = \sum_k p_k \dot{q}_k - L(q_k, \dot{q}_k, t) \quad (2)$$

and Lagrange’s function $L(q_k, \dot{q}_k, t)$ depend on f generalized coordinates q_k ($k = 1, 2, \dots, f = 3N - n$), f canonically conjugate generalized momenta p_k , and f related generalized velocities \dot{q}_k as well as, possibly, explicitly on time t . The degree of freedom f for a system of N point masses consists of the three lateral degrees of freedom for each individual point mass minus the n constraints imposed on the system.

The canonical equations represent the simplest form of equations of motion. Once Hamilton’s function is known the generalized variables of position and canonically conjugate momentum can be obtained by integration.

The total time derivative of an arbitrary function $F(p_k, q_k, t)$, which depends on the same variables and, possibly, explicitly on time, is often called the equation of motion of $F(p_k, q_k, t)$. With Hamilton’s equations from Eq. (1) inserted this equation of motion can partially be expressed by what is called Poisson’s bracket

$$\frac{d F(p_k, q_k, t)}{dt} = \frac{\partial F(p_k, q_k, t)}{\partial t} + \{ F, H \} \quad (3)$$

where Poisson's bracket is given by

$$\{F, H\} = \sum_k^f \left\{ \frac{\partial F(p_k, q_k, t)}{\partial q_k} \times \frac{\partial H(p_k, q_k)}{\partial p_k} - \frac{\partial F(p_k, q_k, t)}{\partial p_k} \times \frac{\partial H(p_k, q_k)}{\partial q_k} \right\}. \quad (4)$$

Poisson's bracket and its quantum mechanical counterpart, the respective commutator, will play an important role further on because if *this arbitrary function does not depend on time explicitly* the equation of motion of $F(p_k, q_k)$ in Eq. (3) is exclusively determined by Poisson's bracket. *If this bracket vanishes $F(p_k, q_k)$ is representing a stationary property.*

2.2 Conservative forces

For conservative forces *Lagrange's function does not depend on time explicitly* and is given by

$$L(q_k, \dot{q}_k) = T(q_k, \dot{q}_k) - V(q_k), \quad (5)$$

i.e., by the difference between the kinetic energy $T(q_k, \dot{q}_k)$ and the potential energy $V(q_k)$. It is also called the kinetic potential.

As a result of Eq. (2), *Hamilton's function as well does not depend on time explicitly* in this case. Additionally, as its Poisson's bracket vanishes because of Eq. (4), Eq. (3) makes it a *constant of the motion*. With Eqs. (2) and (5) we get

$$H(p_k, q_k) = 2T - (T - V) = E, \quad (6)$$

i.e., Eq. (6) expresses the *conservation of the mechanical total energy E* .

As this expression does not include any friction, for instance, this reduced form of the conservation of energy only holds true for a *closed*, purely mechanical system.

Thus, whenever we use such a kind of Hamilton's function we are always dealing with an abstract thought model where we can only consider theoretically how a purely mechanical *closed* system would evolve according to its unperturbed self-dynamics when it would be totally isolated from the outside world. As a consequence, we can only study the behavior of a *closed* model system, which is rather far away from reality. In doing so for a reasonable set of adequate situations we have to choose related initial conditions *a priori*.

2.3 The Hamilton-Jacobi partial differential equation

The integration of the canonical equations of many-particle systems is often tedious. In order to make this task easier one usually tries to find new variables that allow to do the job more efficiently. One such *canonical transformation* of special interest here keeps the original position coordinates but seeks new canonically conjugate momenta with the help of a so-called *shortened* action function $S(q_k, P_k)$, i.e., without explicit dependence on time

$$p_k = \frac{\partial S(q_k, P_k)}{\partial q_k} . \quad (7)$$

Insertion into Eq. (6) brings about the *Hamilton-Jacobi partial differential equation* for conservative systems

$$H \left(\frac{\partial S(q_k, P_k)}{\partial q_k}, q_k \right) = E . \quad (8)$$

2.4 Schrödinger's choice

The Hamilton-Jacobi equation of Eq. (8) is the starting point of Schrödinger's derivation [20, 21, 31]. This derivation is deemed to be well-known. Nevertheless, it seems to us some physical aspects of its result have not been given the necessary attention in the past. For these aspects to unfold we have to take a very close look at what the mathematics are really telling us.

Contrary to the original purpose, it was not a simplification of the integration of Hamilton's equations that Schrödinger had in mind when using the Hamilton-Jacobi equation: Rather, by choosing

$$S(q) = \hbar \ln \psi(q) , \quad (9)$$

i.e, a generating action function proportional to the natural logarithm of an unknown, only position-dependent auxiliary function $\psi(q)$ he was aiming at a totally different target, namely, at finding the extreme values of the *total* energy E by applying a variational procedure to the infinitely extended space integral over this special form of the Hamilton-Jacobi equation

$$\delta J = \delta \int_{-\infty}^{+\infty} \left[H \left(\frac{\hbar}{\psi} \frac{\partial \psi}{\partial q}, q \right) - E \right] d\tau = 0 . \quad (10)$$

Note that the auxiliary function $\psi(q)$ depends on only a single set of position variables, i.e., it applies to a *single* particle.

As we want to work out the classical aspects of Schrödinger's results we do not care about what this auxiliary function $\psi(q)$ might mean physically. Therefore, we shall avoid to call it by its commonly known name. For what we want to show here, all we need to know about $\psi(q)$ are its mathematical properties, especially with respect to an integration over all space.

It is well-known that the hydrogen atom was a hot topic during the first decades of the 20th century. The reason for Schrödinger's choice becomes obvious when we write out his special Hamilton-Jacobi equation for the hydrogen atom (atomic Kepler problem) in full length

$$\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2 - \frac{2m}{\hbar^2} \left(E + \frac{e^2}{r}\right) \psi^2 = 0. \quad (11)$$

The choice in Eq. (9) effects that $\psi(\mathbf{r})$ as well as its first spatial derivatives occur as quadratic forms. Given the unknown properties of the unknown auxiliary function $\psi(\mathbf{r})$ this choice makes sure that nothing is averaged out and lost in the integration over all space to be performed in conjunction with the variational procedure according to Eq. (10).

The purpose of this variational procedure is to determine the unknown auxiliary function $\psi(\mathbf{r})$ such that it makes the variation of the integral vanish in Eq. (10) and to find relative extreme values of the *total* energy E in the *closed system*. Following Schrödinger's original path this demands that

$$\delta J = \delta \int_{-\infty}^{+\infty} d\tau \left[(\nabla\psi)^2 - \frac{2m}{\hbar^2} \left(E + \frac{e^2}{r}\right) \psi^2 \right] = 0. \quad (12)$$

In order to transfer the variation onto $\psi(\mathbf{r})$ and to simultaneously linearize the integrand use is made of the well-known differential vector relation

$$\nabla(\psi\nabla\psi) = (\nabla\psi)^2 + \psi\Delta\psi. \quad (13)$$

This brings about

$$\delta J = \delta \int_{-\infty}^{+\infty} d\tau \left\{ \text{div}(\psi\text{grad}\psi) - \left[\psi\Delta\psi + \frac{2m}{\hbar^2} \left(E + \frac{e^2}{r}\right) \psi^2 \right] \right\} = 0. \quad (14)$$

The volume integral over all space of the partial integrand with the additionally occurring *div* term can be transformed with the help of Gauss's theorem into a surface integral that encloses this infinitely extending volume

$$\delta J = - \int_{-\infty}^{+\infty} d\tau \delta\psi \left[\Delta\psi + \frac{2m}{\hbar^2} \left(E + \frac{e^2}{r}\right) \psi \right] + \oint \delta\psi (\nabla\psi \cdot d\vec{f}) = 0. \quad (15)$$

For this expression to vanish as a whole for any arbitrary variation $\delta\psi$ the unknown auxiliary function $\psi(\mathbf{r})$ has to *meet two demands simultaneously*:

First: The expression in square brackets in the volume integral must vanish, i.e., $\psi(\mathbf{r})$ *must be a solution to Schrödinger's time-independent equation!*

Second: Simultaneously, the surface integral must disappear for any arbitrary variation $\delta\psi$: The directional derivative of $\psi(\mathbf{r})$ in the integrand vanishes with certainty at infinity and independently of any direction if $\psi(\mathbf{r})$ *itself vanishes at infinity*.

If both requirements are met the solutions $\psi(\mathbf{r})$ to Schrödinger's equation *represent* relative extreme values of the mechanical *total* energy [44].

These extreme values belong to extraordinary dynamic situations of a dynamic balance. They are only possible in bound systems. As acts of balance, they must be independent of time and their number must be countable.

This latter requirement is satisfied by the second demand and can equally be met for bound states only: With $\psi(\mathbf{r})$ vanishing at infinity a *boundary value problem* is established for the solutions to Schrödinger's equation [31]. Entailing an eigenvalue problem

$$\mathcal{H}\psi_n(\mathbf{r}) = E_n\psi_n(\mathbf{r}) \quad (16)$$

its eigensolutions $\psi_n(\mathbf{r})$ naturally fulfill the mathematical requirement of countability. The related energy eigenvalues E_n are the relative extreme values of the mechanical *total* energy that have been searched for in the *closed* system. This way, the solutions to Schrödinger's equation establish an *energy representation* of dynamical situations that are extraordinary in a very classical sense as will be shown.

2.4.1 Consequences of an energy representation

From classical mechanics we know: If Hamilton's function does not depend on time explicitly the conservation of the mechanical *total* energy according to Eq. (6) holds true for *any* arbitrary dynamic situation in the *closed* system.

Contrary to the general case, however, the solution $\psi_n(\mathbf{r})$ to Schrödinger's equation selects one of those special dynamic situations for which the *total* energy in the *closed* system has a relative extreme value in terms of the eigenvalue E_n .

Other than the trajectory of a point mass moving in phase-space its *energy representation* is not affected by the uncertainty relations because the relevant combinations of canonically conjugate variables do not occur!

2.5 What does separability mean physically?

In order to show how these extreme values of the mechanical *total* energy come about and what their physical consequences we turn back to the example it all began with [31], the hydrogen problem. For the sake of succinctness, we shall only work out the essential points. For more details we refer to [39]. The hydrogen atom constitutes a spherical problem. With the Laplace operator expressed in spherical coordinates Schrödinger's equation reads

$$\left[-\frac{\hbar^2}{2\mu} \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{\mathcal{L}^2}{2\mu r^2} - \frac{Ze^2}{r} - E \right] \psi(r, \theta, \phi) = 0 \quad (17)$$

where

$$\mathcal{L}^2 = -\hbar^2 \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right]. \quad (18)$$

is the angular dependence given by the square of the operator of the orbital angular momentum. Well-known is the fact that the partial differential equation in Eq. (17) can be separated into an angular and a radial part.

What does separability mean in this case? *It means that only a special dynamic situation can be described because it requires that the energies of the angular and the radial motions be totally independent of each other.* In a spherical potential, there are two ways of how this can happen:

In the first case, the electron would move uniformly along a circular orbit. Such a motion would imply a constant angular kinetic energy with a related stationary orbital angular momentum of the electron and a constant potential energy. Accordingly, we shall encounter these features as we proceed:

The result of the separation procedure is that both parts must equal a constant, the separation constant. How the procedure continues, however, is very important to the physical meaning: In the first step, the separation constant is equated to the differential equation for the square of the operator of the orbital angular momentum. Comparison with its eigenvalue equation identifies the separation constant as the eigenvalue of this operator

$$\mathcal{L}^2 Y_{l,m}(\theta, \phi) = \hbar^2 l(l+1) Y_{l,m}(\theta, \phi) \quad (19)$$

where the $Y_{l,m}(\theta, \phi)$ are the spherical harmonics. Taking the eigenvalue $\hbar^2 l(l+1)$ to replace the separation constant in the differential equation for the energy of the radial motion one obtains in the second step

$$\frac{d^2 u(r)}{dr^2} + \frac{2\mu}{\hbar^2} \left(E + \frac{Ze^2}{r} - \frac{\hbar^2 l(l+1)}{2\mu r^2} \right) u(r) = 0. \quad (20)$$

This equation for the radial energy features a binding potential, the so-called *effective potential*, which is composed of the attractive Coulomb part and the repulsive centrifugal barrier and, therefore, has a minimum for l finite.

Here, it becomes apparent what makes separability necessary: *In the first step a definite orbital angular momentum is determined* as a multiple of \hbar . Of all elliptical orbits with the same angular momentum in a $1/r$ potential only the circular one has extremal total energy.

This is analogous to what Bohr originally only postulated [25, 45–48]. Here, however, his postulate is confirmed in a mathematically sound way. We have to mention, though, that Bohr originally postulated integer *linear* multiples of \hbar . He did so on the basis of planar polar coordinates because, with circular orbits in mind [25, 45], he figured that a stationary orbital angular momentum would define an orbital plane in space. By the way, using this same kind of coordinates in Schrödinger’s equation for the two-dimensional isentropic harmonic oscillator we show in [39] that the same separation procedure results in the eigenvalue equation for the z component \mathcal{L}_z of the orbital angular momentum. Its eigenvalues are integer *linear* multiples of \hbar .

As for the *effective potential*, its minimum is fixed with regard to both its *constant* radius and its value by the orbital angular momentum $l \neq 0$ determined in the first step in accord with Eq. (18)

$$r_{min} = \frac{\hbar^2 l(l+1)}{\mu Z e^2}, \quad (21)$$

$$V_{eff}(r_{min}) = -\frac{\mu Z^2 e^4}{2\hbar^2 l(l+1)} = -\frac{Z^2 e^2}{2a_0 l(l+1)} \quad (22)$$

where a_0 is the Bohr radius.

This procedure constitutes the impression that it is this minimum the solution to Schrödinger’s time-independent equation apparently selects as part of the extremum of the *total* energy. The physical significance of this minimum may be doubted as one might object that *only if an electron had no finite inertial mass it could move uniformly and exactly on the equipotential line at the bottom of the circular valley provided by the minimum of the effective potential. Then its motion would be free of any accelerations.*

Thus, separability of Schrödinger’s *time-independent* equation in spherical coordinates would be the mathematical prerequisite for a uniform and, therefore, *physically stationary* circular motion of a, so-to-say, wave-like electron. As separability excludes any coupling between the energies of the angular and radial motions, it would select dynamic situations where these

energies are totally independent of each other because of the *constant* radius of the minimum of the *effective potential*. This would not allow any radial oscillations that could entail a periodic radial exchange between the constituent energies. No exchange means no periodic accelerations of the electron and, hence, no emission of radiation in *excited* stationary states.

However, any objection to above scenario is unfounded: Although an electron's *finite inertial mass* would always try to keep the electron moving straight and, thus, would lead to a wobbling motion in this circular valley it has to be stated that, given the inertial masses of electron and proton, its coupling to the electromagnetic field is 39 orders of magnitude stronger than its coupling to the gravitational field making the latter interaction totally negligible in the atomic realm.

On the other hand, the notion of being capable of preparing a dynamic balance by making the electron occupy exactly this kind of stationary state is far from realistic as any such attempt would always involve a finite spread of energy. Hence, at best, the motion of the electron will not be exactly circular but superposed by extremely small radial oscillations.

Nonetheless, the truth is we cannot tell from the results of Schrödinger's equation whether it is this behavior or another because the eigenenergies of states with angular momentum as delivered by this equation must not coincide with the energy at the bottom of the respective effective potential but have to be somewhat less deeply bound [49, p188] as they also have to incorporate the energy of the angular motion in order to represent the total energy.

S states with $l = 0$ constitute the *second case* where the separation of the spherical problem makes the energies of the angular and radial motions independent of each other. With only radial motions possible the S states cannot represent a dynamically stationary physical situation even in a strictly closed system because, from a classical viewpoint, an attractive, all the way *pure* Coulomb potential forces the electron into a continuously accelerated radial motion toward a point-like nucleus: The continuously growing nonrelativistic kinetic energy tends toward $+\infty$ while simultaneously the potential energy tends toward $-\infty$ in a way that their sum, the *total* energy, remains a constant relative extremum because generally for $V(r) = +qQ/r$

$$+\frac{dT(r)}{dt} = \dot{r} \cdot m\ddot{r} = \dot{r} \frac{qQ}{r^2} = -\frac{dV(r)}{dt} \quad (23)$$

Here, we already see a shortcoming of an *energy representation* based on the conservation of the *total* energy in closed systems, namely, that such a

representation neither cares about whether an individual dynamic situation is physically reasonable or even possible nor whether an individual dynamic situation differs from other ones if only all have the same *total* energy. Degeneracy is the key feature in this respect, as we shall see further below.

As for the experimentally verified *S* states, their existence is evidence for the fact that the nucleus is not point-like in the mathematical sense, which the pure Coulomb potential makes believe, but has a finite volume, though about 15 orders of magnitude smaller than the one where the electron moves. As the nucleus affects only a minute part of an *s* electron's trajectory it has only a very subtle effect on its energy. The volume effect of isotopic line shifts is the keyword in this respect. More details are discussed in [39].

Just as Bohr [45,48] originally saw it, the stationary circular motion of the first case would be the result of a dynamic balance between the centrifugal and the centripetal forces. As indicated above, however, a more realistic result is an only quasi-stationary situation. Discussed in [39] it is prone to spontaneous transitions. A counterexample is an elliptical orbit where angular and radial motions are coupled and, therefore, not separable. It would always show periodic accelerations.

From all what we have seen so far it is clear that there can only be *one* possible dynamic balance at a time. Therefore, although mathematically tempting because of the linearity of Schrödinger's equation it does not make any sense physically to apply the expansion to the eigensolutions of Schrödinger's *time-independent* equation. Nevertheless, it has been done since. And it is still causing much irritation! To make ends meet one has resorted to the probability interpretation of the absolute squares of the mixing coefficients [13, 14, 20, 22, 29]. This interpretation in terms of a "catalog of possible results" entailed the measurement problem [3] with the related necessity of a "collapse" of the *time-independent* state vector. In order to mitigate the problems related to that collapse the concept of environmentally induced decoherence has been brought into play since the early 1980's [50–52]. Even today, many unintelligible results of quantum mechanics can directly be traced to this practically exclusively accepted interpretation.

All these wiggle-room pitfalls disappear, as we shall see, once we apply, contrary to Schrödinger's original notion [35,36], his *time-dependent* equation to *single closed systems*. Application of the expansion with its *time-dependent* eigensolutions generates *time-dependent* state vectors [12,36]. They lead to *time-dependent* general expectation values, which behave classically.

3 Schrödinger's *time-dependent* equation

If we compare Schrödinger's *boundary value problem* with the one for a clamped membrane we see that his *time-independent* solutions *represent* what would correspond to the node lines of the vibrating membrane but leave its other parts unconsidered. For changing that we now turn to Schrödinger's *time-dependent* equation: In his forth communication [35] he extended his theory to nonconservative systems by allowing of an additional small time-dependent perturbation term in the Hamiltonian as present, e.g., during a radiative transition. For being able to apply time-dependent perturbation theory he had to find a way of how to introduce the time dependence in $\psi(\mathbf{r}, t)$ without violating his previous findings for the unperturbed case. He succeeded by comparing his results with an equivalent of an ordinary wave equation and obtained his *time-dependent* equation this way

$$\mathcal{H}\psi(\mathbf{r}, t) = i\hbar \frac{\partial}{\partial t}\psi(\mathbf{r}, t) . \quad (24)$$

Following Schrödinger the time dependence in the auxiliary function has necessarily to be chosen in terms of a separable energy phase

$$\psi(\mathbf{r}, t) = \phi(\mathbf{r}) e^{-i\frac{E}{\hbar}t} \quad (25)$$

in order to re-establish, when inserted into Eq. (24), the original eigenvalue equation Eq. (16) for the space-dependent part

$$\mathcal{H}\phi_n(\mathbf{r}) = E_n\phi_n(\mathbf{r}) . \quad (26)$$

Its linearly independent eigensolutions $\phi_n(\mathbf{r})$ constitute a complete set of orthonormalized eigenfunctions. Owing to the way they have been constructed in Eq. (25) the $\psi_n(\mathbf{r}, t)$ equally constitute a complete set of orthonormal eigenfunctions. As Schrödinger's *time-dependent* equation is a linear partial differential equation with orthonormalized eigensolutions as well, *the expansion procedure is applicable to the solutions $\psi_n(\mathbf{r}, t)$* . Schrödinger [36] did so for a conservative system as well but only in terms of a time-dependent perturbation approach: Any change with time of an originally stationary state has to be attributed to the action of a small but constant perturbation term.

Here, however, in order to work out the unperturbed internal dynamics in nonstationary states we shall always consider single closed systems only: Hence, Hamilton's operator in Eq. (24) does not dependent on time explicitly.

As a consequence, the conservation of the mechanical total energy holds true as the time dependence exclusively pertains to the auxiliary function $\psi(\mathbf{r}, t)$. This is our most important conclusion with regard to the descriptiveness of the inner dynamics of *single closed systems* in terms of *general* expectation values, albeit from a point of view directly related to energy.

3.1 Expansion in terms of the time-dependent eigenvectors $\psi_n(\mathbf{r}, t)$

Like for the Cartesian unit vectors in Euclidean space, *any* arbitrary linear combination of the $\psi_n(\mathbf{r}, t)$ represents, from an energy point of view, the general dynamics in one of infinitely many nonstationary states in Hilbert space. We shall designate it a normalized *general* state vector not seen this way by Schrödinger [36].

$$\Psi(\mathbf{r}, t) = \sum_n a_n \psi_n(\mathbf{r}, t) = \sum_n a_n \phi_n(\mathbf{r}) e^{-i \frac{E_n}{\hbar} t} \quad (27)$$

$$\text{where} \quad \sum_n |a_n|^2 = 1. \quad (28)$$

Contrary to the state vectors usually found in textbooks, we shall be dealing here with the *time-dependent* ones of Eq. (27). They represent general, i.e., usually nonstationary dynamic situations but, nevertheless, always conserve the mechanical *total* energy of the *closed* system, as we shall see.

Moving in a *dynamic* Hilbert space, this *general* state vector is not “a catalog of possible experimental results” as the mixing coefficients a_n are sometimes referred to by the prevailing probability interpretation [53].

To the contrary, for *theoretically* describing various nonstationary situations in our abstract model system the mixing coefficients $0 \leq |a_n| < 1$ are to be given in advance as the initial conditions. Then, they determine in a *parametrically continuous way* not only the mechanical *total* energy of the dynamics represented by respective nonstationary states in the *closed* system but also the amount of periodic exchange between constituent energies.

Such an exchange is characteristic of the dynamics in nonstationary states of *closed* systems. This exchange is not caused by perturbations but by the system’s unbalanced inner dynamics. For demonstrating how this exchange comes about we first have to introduce the *general* expectation value.

3.2 The *general* expectation value of an arbitrary operator

The *general* expectation value of an operator \mathcal{O} describes the time dependence of a property \mathcal{O} also in nonstationary states. The characterization as *general* refers to its generic state vector, the time-dependent *general* state vector of Eq. (27). We define the *general* expectation value by

$$\begin{aligned}\langle \mathcal{O} \rangle &= \langle \Psi | \mathcal{O} | \Psi \rangle = \sum_m \sum_n a_m^* a_n \langle \psi_m | \mathcal{O} | \psi_n \rangle \\ &= \sum_m \sum_n a_m^* a_n e^{i \frac{E_m - E_n}{\hbar} t} \langle \phi_m | \mathcal{O} | \phi_n \rangle.\end{aligned}\quad (29)$$

It is clear only the mathematical properties of ϕ_m and ϕ_n with regard to an integration over all space matter for the matrix elements $\langle \phi_m | \mathcal{O} | \phi_n \rangle$. The first total time derivative of the *general* expectation value is obtained [54] as

$$\begin{aligned}\frac{d\langle \mathcal{O} \rangle}{dt} &= \sum_m \sum_n a_m^* a_n \frac{i}{\hbar} (E_m - E_n) e^{i \frac{E_m - E_n}{\hbar} t} \langle \phi_m | \mathcal{O} | \phi_n \rangle \\ &+ \sum_m \sum_n a_m^* a_n e^{i \frac{E_m - E_n}{\hbar} t} \left\langle \phi_m \left| \frac{\partial \mathcal{O}}{\partial t} \right| \phi_n \right\rangle,\end{aligned}\quad (30a)$$

$$\frac{d\langle \mathcal{O} \rangle}{dt} = \frac{1}{i\hbar} \langle \Psi | [\mathcal{O}, \mathcal{H}] | \Psi \rangle + \left\langle \Psi \left| \frac{\partial \mathcal{O}}{\partial t} \right| \Psi \right\rangle.\quad (30b)$$

Comparison with Eqs. (3) and (4) shows that above equation is totally analogous to the classical expression because the properties of the commutator exactly correspond to those of Poisson's bracket. This corroborates our claim that the expansion in terms of eigenvectors is physically meaningful only in conjunction with the *time-dependent* eigensolutions to Schrödinger's *time-dependent* equation: Only then it can reproduce the classical equivalent.

The way we stipulated to use Schrödinger's equations only for *closed systems* admits only *not explicitly time-dependent* operators because the mechanical *total* energy has to be conserved. Thus, having to leave the second term on the right hand side unconsidered only the description of *closed systems* is possible, and the equation of motion of the operator \mathcal{O} in Eq. (30b) is totally determined by the commutator between \mathcal{O} and the Hamiltonian \mathcal{H} leaving only two possibilities: Either \mathcal{O} commutes with \mathcal{H} or it does not.

3.2.1 \mathcal{O} commutes with \mathcal{H}

In this first case, \mathcal{O} has the same eigenfunctions as \mathcal{H} . Then, its matrix in Eq. (29) has only diagonal elements and its *general* expectation value reduces to a weighted sum of its eigenvalues

$$\langle \Psi | \mathcal{O} | \Psi \rangle = \sum_n |a_n|^2 O_n . \quad (31)$$

Hence, the *general* expectation value of the operator \mathcal{O} here describes properties that are always stationary independently of the actual dynamic situation.

Commutativity holds true especially for the Hamiltonian \mathcal{H} itself. This way, it makes sure that the mechanical *total* energy is conserved even if the *closed* system is *not* in a stationary state

$$\langle \Psi | \mathcal{H} | \Psi \rangle = \sum_n |a_n|^2 E_n = E_{tot} . \quad (32)$$

Provided the condition of normalization in Eq. (28) is met the sum in above equation may be equal for different weights and energy eigenvalues. This means different configurations in different nonstationary states, i.e., different dynamic situations may have the same mechanical *total* energy. Hence, *degeneracy is an inherent feature of an energy representation of nonstationary states in a closed system.*

Noting that the expression in Eq. (32) is identical to the one propagated in textbooks and obtained by using the usual, time-independent state vectors, we find confirmed our claim that a “collapse” of the general state vector is not a reasonable physical concept because it *would violate the conservation of the mechanical total energy in nonstationary states of the closed system.*

Hence, there are no “many worlds” [55, 56] and no “parallel universes”! Moreover, there is no measurement problem either because it is the time average for a single particle and not an ensemble average that would be measured. Also, an abstract *closed* system cannot be interfered with experimentally. For making the theoretical description of a real experiment possible one would have to open the system and include the environment by implementing all the necessary mathematical features in the Hamiltonian.

Concluding this first case we find that Einstein was right when he insisted on “God does not toss dice!” [37].

3.2.2 \mathcal{O} does not commute with \mathcal{H}

In this second case, when the operator \mathcal{O} does not commute with the Hamiltonian \mathcal{H} the time-dependent *general* expectation value of \mathcal{O} describes truly dynamic properties in nonstationary states of the *closed* system.

We have seen in section 3.1 that the expansion procedure composes a *general* state vector of a countable number of time-dependent eigensolutions $\psi_n(\mathbf{r}, t)$. As a result, the indices m and n in the double sum in Eq. (29) run through the same set of natural numbers. This kind of countability effects that always only square matrices occur in a *general* expectation value. These square matrices have the same number of nondiagonal elements above and below the main diagonal. This enables us to split this double sum into three parts according to whether $m = n$, $m > n$, and $m < n$

$$\begin{aligned} \langle \mathcal{O} \rangle(t) = \langle \Psi(t) | \mathcal{O} | \Psi(t) \rangle &= \sum_n |a_n|^2 O_n \\ &+ \sum_{m>n} a_m^* a_n e^{+i \frac{E_m - E_n}{\hbar} t} \langle \phi_m | \mathcal{O} | \phi_n \rangle \\ &+ \sum_{m<n} a_m a_n^* e^{-i \frac{E_m - E_n}{\hbar} t} \langle \phi_n | \mathcal{O} | \phi_m \rangle . \end{aligned} \quad (33)$$

Provided \mathcal{O} is an Hermitean, i.e., self-adjunct operator its off-diagonal matrix elements are related by

$$\langle \phi_n | \mathcal{O} | \phi_m \rangle = \langle \phi_m | \mathcal{O} | \phi_n \rangle^* , \quad (34)$$

i.e., a transposed element equals the complex conjugate of the original element. This makes the two lower sums in Eq. (33) complex conjugate to each other. They add to obtain twice the real part

$$\begin{aligned} \langle \mathcal{O} \rangle(t) = \langle \Psi(t) | \mathcal{O} | \Psi(t) \rangle &= \sum_n |a_n|^2 O_n \\ &+ \sum_{m>n} 2 \Re \left\{ a_m^* a_n \langle \phi_m | \mathcal{O} | \phi_n \rangle \right\} \cos \left(\frac{E_m - E_n}{\hbar} t \right) \\ &- \sum_{m>n} 2 \Im \left\{ a_m^* a_n \langle \phi_m | \mathcal{O} | \phi_n \rangle \right\} \sin \left(\frac{E_m - E_n}{\hbar} t \right) . \end{aligned} \quad (35)$$

This result demonstrates that the hermiticity of the matrix elements is a stringent prerequisite for the ability to describe multiple periodic phenomena

in *closed* systems the eigenfrequencies of which conform to how they were originally postulated by Bohr [1, 2, 8, 9, 25–27, 45–48, 57]

$$\omega_{m,n} = \frac{E_m - E_n}{\hbar} . \quad (36)$$

In this context, we have to mention that Schrödinger was strongly influenced by de Broglie’s “phase wave” concept [58, 59], which he saw confirmed by the undulatory character of the solutions to his *time-independent* equation and, therefore, called them wave functions. All through the years, he kept interpreting the *time-independent* spatial part of the eigenfunctions of the stationary states as expressing proper vibrations in terms of standing waves and the eigenenergies as equivalents of the eigenfrequencies of these proper vibrations [7, 13, 26, 27, 29, 31, 36, 37]. Transition frequencies, in his view, were differences of these eigenfrequencies.

Because of energy conservation the eigenfrequencies in Eq. (36), in our view the real ones, would not be observable outside a *closed* system. Their simultaneous occurrence, however, is a characteristic feature of any boundary value problem not only in *closed* systems: Just think of the overtones of a clamped string.

All eigenfrequencies depend on the actual potential in the Hamiltonian, which Schrödinger’s equation has been solved for. This is why a general multiple periodic motion can usually not be expanded into a simple Fourier series.

The number of active eigenfrequencies depends on the number of nonzero matrix elements as well as on the given number of nonzero mixing coefficients. According to Eq. (32) these nonzero coefficients also determine the mechanical *total* energy of any dynamic situation in the *closed* system.

This way the nonzero mixing coefficients act as initial conditions in the abstract thought model of a *closed* system. In such a system these coefficients have to be constant in order to conserve the mechanical *total* energy no matter what the dynamic state. Thus, as we have seen in Eq. (32), as long as the system is to be considered *closed* any dynamic situation is energetically stable. As a *closed* system its dynamics cannot be affected by external perturbations.

However, if we allow the system to become only *quasi-closed* and, so, somewhat more realistic by relaxing the strict demand for this constancy and by admitting a very slow time dependence (slow when compared with the longest period of the inner oscillations) of the mixing coefficients we can

describe, e.g., the time evolution of a *complete* spontaneous transition as being driven by unbalanced internal dynamics. Details may be found in [39]. There it even turns out that this process becomes the more “jump”-like, i.e., the more instantaneous-like in the time domain, the higher the transition probability.

3.3 For example: Hydrogen

Application of Eq. (35) to the atomic Kepler problem (hydrogen) obtains for the *general* expectation value of the nonstationary *radial* kinetic energy [39]

$$\begin{aligned}
\langle \mathcal{H}_{kin} \rangle(t) &= + \sum_{n=1} \sum_{l=0}^{n-1} \sum_{m=-l}^{+l} |a_{n,l,m}|^2 \left(E_n + \frac{Z^2 e^2}{a_o n^2} \right) \\
&+ 2 \sum_{k>n=1} \sum_{l=0}^{n-1} \sum_{m=-l}^{+l} \left\langle R_{k,l}(r) \left| \frac{Ze^2}{\hat{r}} \right| R_{n,l}(r) \right\rangle \Re \left\{ a_{k,l,m}^* a_{n,l,m} \right\} \cos \omega_{k,n} t \\
&- 2 \sum_{k>n=1} \sum_{l=0}^{n-1} \sum_{m=-l}^{+l} \left\langle R_{k,l}(r) \left| \frac{Ze^2}{\hat{r}} \right| R_{n,l}(r) \right\rangle \Im \left\{ a_{k,l,m}^* a_{n,l,m} \right\} \sin \omega_{k,n} t
\end{aligned} \tag{37}$$

and for the nonstationary *radial* potential energy [39]

$$\begin{aligned}
\langle \mathcal{V} \rangle(t) &= - \sum_{n=1} \sum_{l=0}^{n-1} \sum_{m=-l}^{+l} |a_{n,l,m}|^2 \frac{Z^2 e^2}{a_o n^2} \\
&- 2 \sum_{k>n=1} \sum_{l=0}^{n-1} \sum_{m=-l}^{+l} \left\langle R_{k,l}(r) \left| \frac{Ze^2}{\hat{r}} \right| R_{n,l}(r) \right\rangle \Re \left\{ a_{k,l,m}^* a_{n,l,m} \right\} \cos \omega_{k,n} t \\
&+ 2 \sum_{k>n=1} \sum_{l=0}^{n-1} \sum_{m=-l}^{+l} \left\langle R_{k,l}(r) \left| \frac{Ze^2}{\hat{r}} \right| R_{n,l}(r) \right\rangle \Im \left\{ a_{k,l,m}^* a_{n,l,m} \right\} \sin \omega_{k,n} t
\end{aligned} \tag{38}$$

where, of course, $E_n = -\frac{Z^2 e^2}{2a_o n^2}$. The constant parts in the first line in each of these equations also embrace the constant energy of the underlying angular motion. For details check with [39].

The multiple periodic *radial* exchange between the constituent energies is obvious. The reason for this kind of its occurrence is related to the fact that the electron moves in a highly anharmonic potential:

Mathematically, this behavior is being accounted for by how the eigenfrequencies here originate from the structure of the energy eigenvalues: The

higher the electron moves in the potential, i.e., the higher the available energy, the lower frequencies have to be added because the farther the electron can move out into the shallower upper part of the Coulomb potential, the slower becomes its motion there. In the simplest case the electron's orbit is an ellipse with only *one* frequency.

Connected with this exchange is the conservation of the mechanical *total* energy

$$E_{tot} = \langle \mathcal{H}_{kin} \rangle(t) + \langle \mathcal{V} \rangle(t) = \sum_{n=1}^{n-1} \sum_{l=0}^{+l} \sum_{m=-l}^{+l} |a_{n,l,m}|^2 E_n = -\{ \langle \mathcal{H}_{kin} \rangle(t) \}_t \quad (39)$$

and that the *time averages* of the *general* expectation values of these energies follow the classically well-known *Virial theorem*

$$2\{ \langle \mathcal{H}_{kin} \rangle(t) \}_t = -\{ \langle \mathcal{V} \rangle(t) \}_t = \sum_{n=1}^{n-1} \sum_{l=0}^{+l} \sum_{m=-l}^{+l} |a_{n,l,m}|^2 \frac{Z^2 e^2}{a_o n^2}, \quad (40)$$

which holds for *any* dynamic situation in the *closed* system.

The conservation of the mechanical *total* energy in such a system is also the reason why the frequencies of the very complex *radial* oscillations, which effect said exchange of the electron's energies, would only be perceptible inside the *closed* system by somebody who would hypothetically be able to watch the electron. The reason being that the conservation of the mechanical *total* energy is also responsible for the conservation of the total orbital angular momentum by only allowing *radial* dependences for the operators involved. This way, the orthonormality of the angular momentum eigenfunctions accounts for the constancy of the orbital angular momentum by reducing the summations in Eqs (37) and (38) to the indices given there.

This is the second reason why these oscillations would not be capable of generating electromagnetic radiation despite the degeneracy of the energy levels with respect to angular momentum quantum numbers. Thus, the *closedness* of the system is the limiting factor for any further knowledge that goes beyond these constraints.

Nevertheless, it shows us that the inner dynamics must not be left aside when it comes to understand how, for instance, a *spontaneous* electromagnetic transition evolves with time. For details we refer to Part II in [39]. In this context, it proves fortunate that this process can be treated on the basis of a balance between the decrease of the atomic *total* energy per unit of time

and the power of the emitted electromagnetic radiation, i.e., so-to-say, on the basis of an *energy representation* per unit of time. However, an essential prerequisite of the approximation used is the fact that the inner dynamics and the instantaneous loss of energy due to *spontaneous* emission can be treated on totally different time scales that are so far apart that the Hamiltonian may still be considered not dependent on time explicitly. This is not possible for induced absorption or emission. Therefore, as these processes would require an explicitly time-dependent Hamiltonian, Schrödinger's equation as we apply it here is basically not suited for their dynamic description.

3.4 Further examples

In this section we shall give a short account of further examples details of which may be found in [39, 40]:

3.4.1 Free particles

An *energy representation* of a free particle in a *closed system* need not care about where it is and in which direction it is moving. With a flat potential plane the particle finds the same conditions everywhere. Therefore, it can be anywhere, even at infinity of an infinitely extended *closed* system. As a consequence, an *energy representation* of a free particle *must not comply* with the second requirement of Schrödinger's variational procedure, the vanishing of the auxiliary function at infinity. Hence, without having to meet the boundary conditions the auxiliary functions need not be square integrable and orthonormal because there is nothing to be linearly superposed: Each particle keeps the initially given momentum once and for all because, due to commuting with the Hamiltonian, the momentum is a constant of the motion as well. Especially the inclusion of the time dependence shows that wave packets would violate this property. But there is no need of localization as we elaborated on above: Each particle is in a stationary state for any initially assumed nonrelativistic energy. The *representation* of their individual *energetic* situation by a plane wave, which is infinitely extended in all directions and characterized by a single wave number, is appropriate for a single particle for above given reasons and not in contradiction to Schrödinger's equation because there are no relative extrema of the *total* energy that would have to be counted. Therefore, from this point of view, there is no need to infer from this *representation* of its *total* energy wave properties for the particle itself

how it is being done, e.g., for the interpretation of scattering experiments. On the other hand, however, the result for the *total* energy suggests for the particle's momentum the relation $p = \hbar k$ and, therefore, wave properties.

3.4.2 The linear harmonic oscillator

The linear harmonic oscillator is one of the most irritating examples that most textbooks would not want to miss dealing with. Stationary states in such an oscillator that is really active: Classically incomprehensible! The only classically stationary state one may think of as natural exists when the particle is at rest on the bottom of the harmonic potential. This, however, would correspond to the trivial solution $0 = 0$ of $H = E$ but an *energy representation* has to deal with situations where $E \neq 0$.

In this context the question arises why do the solutions to Schrödinger's *time-independent* equation for the linear harmonic oscillator single out situations as stationary that evidently have *total* energies $E_n > 0$? The answer may be as follows: If you do not care about an overall time dependence as existing and only look at the tiny instant in time when the oscillator comes to a total standstill at the turning point where the *total* energy equals the maximum of the potential energy then you may call this instant stationary.

In order to exclude the trivial solution there has to be a minimal amount of energy in the *closed* system, but simultaneously this amount has to comply with the *only* characteristic feature of the given harmonic potential, its eigenfrequency ω . Any linear harmonic oscillator can only admit *one single eigenfrequency* for any nonrelativistic *total* energy! *Mathematically this feature is being accounted for by the equally spaced eigenvalues* $E_n = \hbar\omega(n + \frac{1}{2})$ that belong to the orthonormal eigenfunctions $\psi_n(x, t)$. These eigensolutions must not be mistaken for representing lasting *physically* stationary states: They do not exist for $E \neq 0$! This becomes immediately clear in conjunction with Eq. (35): The *general* expectation values of the kinetic and the potential energy show the familiar and classically well-known exchange behavior with the frequency 2ω as well as the concomitant conservation of the *total* energy while their time averages comply with the *Virial theorem* for a harmonic potential, i.e., kinetic and potential energy have, on time average, equal constant shares in the *total* energy.

The uncertainty relation is held responsible for what is being called “zero-point” energy, namely, said minimal amount of energy: It is interesting to see [39] that the *general* expectation value of the commutator between \hat{x}

and \hat{p}_x reproduces the well-known relation $\langle[\hat{x}, \hat{p}_x]_-(t)\rangle = i\hbar$ for *any* non-stationary dynamic situation while for the respective anti-commutator the first time derivative of the *general* expectation value of the potential energy is obtained $\frac{1}{2}\langle[\hat{x}, \hat{p}_x]_+(t)\rangle = \frac{m}{2}\frac{d}{dt}\langle\hat{x}^2\rangle(t)$ in accord with the classical relation $xp_x = \frac{m}{2}\frac{d}{dt}(x^2)$. With the help of the definitions of the creation and annihilation operators as well as their commutator relation an equivalent expression can be obtained for the operators alone, namely, $[\hat{x}, \hat{p}_x]_+ = m\frac{d}{dt}\hat{x}^2$ and, in the same way, also the classical equation of motion for the oscillator: $\frac{d^2}{dt^2}\hat{x} + \omega^2\hat{x} = 0$. Additionally, as neither \hat{x} nor \hat{p}_x commute with the Hamiltonian \mathcal{H} the linear harmonic oscillator must never stop oscillating! For this to hold it is mandatory that there always be a constant amount of energy in the *closed* system and, so, provides further evidence for the *mathematical* necessity of the so-called “zero-point” energy. This minimal amount of energy, however, is not enforced by uncertainty relations but by the way of how the *physical* situation is being described mathematically by an *energy representation* that here must always pertain to *nonstationary situations*.

3.4.3 The two-dimensional isotropic harmonic oscillator

The two-dimensional isotropic harmonic oscillator features the same dynamic behavior as the linear one not only with regard to energy exchange, conservation of the *total* energy, and the *Virial theorem*. In spite of possessing an orbital angular momentum it is equally not capable of holding *physically* stationary states. This results from how the equidistant energy eigenvalues $E_n = \hbar\omega(2N + 1 + M) = \hbar\omega(n + 1)$ feature a never vanishing “1”, which has to be attributed to the linear part of a particle’s motion: In this kind of harmonic potential the motion of the particle is confined to a plane. Its orbital angular momentum $\hbar M$ characterized by the azimuthal quantum number M produces a centrifugal barrier, which keeps the particle automatically away from the origin. Therefore, the motion of the particle in this potential consists of a superposition of a uniform circular motion and a linear oscillation superimposed on the former. Both types contribute to the periodic energy exchange with the radial oscillation frequency 2ω . In three dimensions, this is what happens to an ion in a Paul trap [60].

3.4.4 The time dependence of spontaneous multipole transitions

Using a two-level model the power balance between the loss of atomic energy in nonstationary states and the emitted radiative power provides a simple *nonlinear* differential equation for the time dependence of the mixing coefficients. This time dependence is capable of accounting for all features of the light emitted in a *complete* spontaneous transition, and all this without involving QED in the first place: The unpredictability of its occurrence in a single atom is caused by the uncertainties in the preparation process and related to the dwell time in a quasi-stationary situation that is followed by a well-defined “jump time”, which also entails a corresponding coherence length of the emitted pulse. The Fourier transform of its time-dependent intensity distribution is dominated by a Lorentzian profile in the frequency domain. The characteristic features of such a single emission process resolve in a very natural way the conundrum of the alleged wave-particle duality of what according to Einstein’s rather diffuse notion is since being called “photon”: wave-like at low frequencies, particle-like at very high ones. The knowledge of the individual process also makes it possible to understand why a bulk of simultaneously excited atoms decays exponentially.

QED claims that only the quantization of the electromagnetic field makes the spontaneous emission process intelligible as being induced by fluctuations of the photon vacuum. This notion is related to the asymmetry of the action of the creation and annihilation operators with regard to photon numbers. Our model, however, shows that in a *quasi-closed* system these fluctuations might at best contribute to this process by preventing any potentially truly stationary state from surviving any longer period of time.

3.4.5 Angular momentum coupling in arbitrarily oriented magnetic fields

In most cases, an *energy representation* has the disadvantage that because of the inherent degeneracy only indirect conclusions about the dynamic behavior of the particle in the *closed* system are possible. This shortcoming does not matter very much as long as well-known classical problems are being dealt with. Beyond these cases it becomes problematic and takes some fantasy. But there are exceptions. They are related to the rare cases where a relevant property occurs *linearly* in the Hamiltonian. Even then it takes the simplest of cases to allow an unambiguous and classically illustrative

description of what is happening.

Such a case is the angular momentum coupling in an arbitrarily oriented magnetic field when the two angular momenta are associated with spin $\frac{1}{2}\hbar$ each but with *different* magnetic moments as, for instance, for the hyperfine coupling in silver. The result of the quite straightforward but very voluminous calculations [40] is that the *general* expectation value of the resultant *total* magnetic moment shows in the general nonstationary case a *three-dimensional* motion consisting of a Lamor precession about the direction of the external magnetic field superimposed on which are one longitudinal and four transverse oscillation modes in accord with the eigenfrequencies that the prevalent interpretation would relate to magnetic dipole (M1) “transitions” with $\Delta M_F = 0$ and $\Delta M_F = \pm 1$, respectively, in the scheme of field-dependent energy eigenvalues. These oscillation frequencies originate from a very complex oscillation of the coupling angle that results from unbalanced, on the one hand, mutually inflicted internal torques by the coupling magnetic moments and, on the other hand, those imposed on both of them by the external magnetic field. The resultant nonstationary dynamics are responsible for the multi-frequency periodic changes of the length of the *total* magnetic moment that go along with this oscillation of the coupling angle. This behavior also explains how an atomic magnetic “antenna” works and how the respective polarizations of the emitted M1 radiations come about.

The longitudinal oscillation mode is additionally responsible for a characteristic feature of nonstationary states in a *closed* system: The periodic exchange between constituent energies, here between the energy of angular momentum coupling and the *Zeeman* energy. The reason for the involvement of only the longitudinal oscillation mode is simple: The scalar product only makes the longitudinal component of the *total* magnetic moment relevant to the *Zeeman* energy.

If the vector of the *total* magnetic moment itself would be parallel to the direction of the external magnetic field there would be no Lamor precession because the magnetic field cannot exert a torque on the *total* magnetic moment. Notwithstanding this fact, there would still be a longitudinal oscillation of the length of the *total* magnetic moment because the internal and external torques on the *different* constituent magnetic moments are not balanced in the general case.

This result could not have been obtained on the basis of the “eigenstates-only” doctrine. For a spin $\frac{1}{2}\hbar$ it mandates that this spin only point either “up” or “down” with respect to the field direction. This is how the result of

the Stern-Gerlach experiment is being interpreted until this very day.

As for the interpretation of this experiment we find for the overwhelming number of initial conditions that what starts at low field strengths as a non-stationary situation of an *induced total magnetic moment* may end up as an at least quasi-stationary one:

This behavior is owed to the *difference* of the force acting on the *different* constituents in the strong gradient in the stray-field zone before the atoms enter the gap between the artificially crafted pole pieces. In a strongly nonlinear feedback process the *different* pushes and pulls there make the composition of the components of the eigenvectors change. Accompanied by an energy transfer to or from the kinetic energy this causes the inner energy to change toward one of the two at least quasi-stationary situations. This naturally explains the experimental result [40].

This example also demonstrates how the energy eigenvalues represent situations of dynamic balance: Whatever the direction of the field, in order to be stationary the vector of the resultant *total* magnetic moment must, first, be parallel to the field direction so that the field cannot exert a torque on the *total* magnetic moment.

However, second, as long as the internal torques that the *different* coupling magnetic moments mutually exert on each other and the *different* external torques inflicted by the field on them individually are not balanced the length of the *induced* total magnetic moment would still keep oscillating longitudinally irrespective of any decoupling that takes place in high fields.

In a *closed* system, all oscillation modes related to the coupling angle are only absent and the balance of all torques is achieved if the coupling of the constituent magnetic moments is prepared *a priori* with a specific, field-dependent coupling angle. These coupling angles are responsible for the anomalous *Zeeman* effect as they provide a field-dependent length of the vector of the stationary *total* magnetic moment and, thereby, effect the nonlinearity of the field dependence of the energy eigenvalues in the nonlinear *Zeeman* regime. Thus, they make this effect a matter of purely magneto-mechanical origin. In the field-free case, the Clebsch-Gordan coefficients represent the coupling angles that make the system stationary.

4 Conclusions

In section 2.4 we have reproduced how Schrödinger's equation emerged from a variational procedure. But on the first page of his second communication [32] Schrödinger was reasoning about the way he obtained his results in his first communication [31] and admitted that his course of action there was neither intuitive and nor easy to understand. In a footnote on the same page of [32] he therefore declared that he would not follow that course further on. He would instead perceive Hamilton's principle of variation as equivalent to Fermat's principle of wave propagation in configuration space.

At this point we have to explain that Schrödinger was so much devoted to his wave interpretation that he never alluded to *relative extreme values of the total energy* in the *closed* system as being selected by the result of his variational procedure nor did he do so to the circumstance that these results could represent extraordinary situations of dynamical balance. *This is our interpretation.* However, given the many examples we have calculated in terms of *general* expectation values and presented here as short excerpts we are convinced that our interpretation is physically the most plausible one.

As long as we are dealing with Hamiltonians that do not depend on time explicitly, we are only able to describe *closed* systems, which evolve with time *deterministically* according to their intrinsic dynamics, and, as we have shown, whose *classically intelligible dynamic properties are equally well accounted for by the quantum mechanical formalism*, at least as far as it is possible for an *energy representation*. This confirms their common classical roots. Apart from degeneracies, varying settings, i.e., varying initial conditions render the concomitant compositions of the *general* state vector capable of *continuously* covering all energies and other potentially observable properties, however, for a *closed* system only in a *parametrically continuous* way. Hence, as long as the system is considered *closed* and, therefore, theoretically not allowed to communicate its inner state, e.g., by voluntarily giving away part of its energy in terms of *spontaneously* emitted radiation, a passive hypothetical observer cannot tell from outside what is going on inside.

A *closed* system is a very abstract entity without a real existence, indeed. However, if after interference from outside with a short preparation process a *real* system finds itself in a situation where environmental influences are so strongly reduced that they can only have a negligible impact we may consider this a good experimental approximation of what we may at least call a *quasi-closed* one. A real example would be an excited atom or molecule in

a highly evacuated chamber where the mean free path exceeds the physical dimensions of the chamber by orders of magnitude so that the collision rate with rest gas particles can safely be neglected. A shielding against ambient electromagnetic radiation may further help to improve this approximation.

For enabling a passive hypothetical observer to attain more knowledge, we would have to relax the closure condition by demanding that any voluntary temporal change of the total *inner* energy take place so slowly that the instantaneous intrinsic dynamics are left practically unaffected. This keeps the system *quasi-closed* by affecting the system's energy noticeably only after a great many cycles of the multiple periodic motion. Then, the passive observer can get a lot of genuine information about the system and its dynamics like frequency, polarization, and intensity of the *spontaneously* emitted radiation, and all this without actively disturbing it during the emission process.

An experimental realization of this scenario would exist during the period of time after this atom or molecule has been excited by a short laser pulse, i.e., after the initial preparation of the *total* energy in the excited *quasi-closed* system, when the particle is left all by itself. Then we have the situation that its inner nonstationary dynamics evolve according to its almost not explicitly time-dependent Hamiltonian. "Almost not explicitly time-dependent" means that the progress of the *spontaneous* emission process is slow enough and the instantaneous loss of internal energy small enough to leave its inner dynamics adiabatically adjusted. The fingerprint character of the individual spectra is evidence for the validity of this assumption.

That a system's evolution is undisturbed by an observer is all the more true the farther the distance between source and observer. If we take this distance to astronomical scales like five billion light years, the formation of the earth's galaxy may still have been underway, and it was not foreseeable that the evolution would eventually bring about intelligent human beings that some day would be able to measure cosmic radiation. So, the measuring process could neither have interfered with the source process nor could the latter have been influenced by the observer's consciousness [3, 56]. Both processes are totally decoupled in astronomical observations. This casts doubts on the demand that in a quantum mechanical measurement source, radiation, and detector always have to be considered one quantum mechanical system [42]. Although basically correct, this demand would mean that every conceivable interaction in this system would have to be accounted for in the Hamiltonian describing it, a task that can hardly be accomplished.

The fundamental assumption in Bohr's first postulate [25, 45, 46] was

that an atomic system can permanently exist only in a number of stationary states. This postulate has been axiomatically empowered by Born's and his coworkers' [10–12] “eigenstates-only” doctrine that claims physical reality exclusively for stationary states because observable properties are deemed to be related to these states only. Although, as we have shown, this exclusiveness does not even hold for a *closed* system the respective dynamic situation could nevertheless be possible where *physically* permissible. And although extremely unlikely if not even impossible, the exclusiveness of this dictum, however, is the basis for the pretended necessity of state vector collapse. It demands that upon measurement the normalized sum over the absolute squares of the mixing coefficients reduce to just one of them thereby pinpointing exactly *one* final stationary state. For a *real* dynamic system, this is a rather weird kind of assumption because a dynamic balance is extremely difficult if not impossible to prepare. This makes stationary states extraordinary if not even fictitious dynamic situations. An illustrative example is a magnetic moment in a magnetic field: Dynamic balance *and* extremal energies for $\theta \equiv 0$ or π , nonstationary in between. Therefore, the focus should rather be on the nonstationary states as *represented* by *general* state vectors because only *general* expectation values describe the truly *general* dynamic situation in a classically intelligible way, and they do so way beyond the extend possible with any order of time-dependent perturbation theory. This kind of approximation theory is only the walking stick that the accepted doctrine of “stationary states only” has to employ in order to overcome the inherent denial of acknowledging the existence of deterministic dynamic processes.

However, as any time-dependent perturbation theory can only explore the dynamics in the more or less closer vicinity of stationary states the ensuing time dependence is restricted to the very early beginning of, e.g., a *spontaneous* emission process. Providing only transition probabilities, this is all what the application of first-order time-dependent perturbation theory is capable of delivering on the basis of the interpretation that only eigenstates represent physical reality. This way, a very selective perception of the general dynamics and of reality has been raised to the doctrine of seemingly quantal behavior. This would be tantamount to maintaining that only the node lines on a drum hide are physically real but not the sound it makes.

This is not the case if the inherent dynamics are taken into account: Any deviation from the exact point of balance is tantamount to a residual oscillation that makes the *quasi-closed* system prone to *spontaneous* transitions because the concomitant accelerations make the related *effective* potential,

so-to-say, leaky. This is the feature that has been exploited in Part II of [39] for the description of a *complete* spontaneous transition.

Concluding we state: *Schrödinger's derivation of his equations makes it clear that the solutions to his time-independent equation represent special dynamic situations. This property can only be seen in connection with his time-dependent equation and the application of time-dependent state vectors in closed systems. Related time-dependent general expectation values are continuous and do not justify the claim of establishing "new physics". To the contrary, they represent a continuation of classical mechanics from a different point of view, one that rather focuses on energy than on phase space.*

Whether or not the solutions $\psi(\mathbf{r}, t)$, one way or another, have a physical meaning of their own beyond their mathematical properties is not being disputed here. What is seen questionable, however, is the alleged indeterminacy of quantum mechanics as expressed by the probability interpretation [20–22].

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References

- [1] Bohr, N.: Linienspektren und Atombau. Ann. Physik **71**, 228 (1923)
- [2] Bohr, N.: Atomtheorie und Mechanik. Naturwiss. **14**, 1 (1926)
- [3] Neumann, J. von: Mathematische Grundlagen der Quantenmechanik. Springer, Berlin (1932, reprint 1968)
- [4] Fermi, E.: Quantum Theory of Radiation. Rev. Mod. Phys. **4**, 87 (1932)
- [5] Heisenberg, W.: Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik. Z. Phys. **43**, 172 (1927)
- [6] Jordan, P.: Über quantenmechanische Darstellung von Quantensprüngen. Z. Phys. **40**, 661 (1927)

- [7] Schrödinger, E.: Are there Quantum Jumps? Br. J. Philos. Sci. **3**/10, 109; *ibid.* 233 (1952)
- [8] Bohr, N.: Über die Serienspektren der Elemente. Z. Phys. **2**, 423 (1920)
- [9] Bohr, N.: Abhandlungen über Atombau. F. Vieweg & Sohn, Braunschweig (1921)
- [10] Born, M., Jordan, P.: Zur Quantenmechanik. Z. Phys. **34**, 858 (1925)
- [11] Born, M., Heisenberg, W., and Jordan, P.: Zur Quantenmechanik II. Z. Phys. **35**, 557 (1926)
- [12] Born, M.: Das Adiabatenprinzip in der Quantenmechanik. Z. Phys. **40**, 167 (1926)
- [13] Born, M.: Zur Quantenmechanik der Stossvorgänge. Z. Phys. **37**, 863 (1926)
- [14] Dirac, P. A. M.: The Fundamental Equations of Quantum Mechanics. Proc. Roy. Soc. London **109**, 642 (1925)
- [15] Dirac, P. A. M.: On the Theory of Quantum Mechanics. Proc. Roy. Soc. London **A112**, 661 (1926)
- [16] Dirac, P. A. M.: The Physical Interpretation of Quantum Dynamics. Proc. Roy. Soc. London **A113**, 621 (1927)
- [17] Heisenberg, W.: Quantenmechanik. Naturwiss. **14**, 989 (1926)
- [18] Heisenberg, W.: Schwankungserscheinungen und Quantenmechanik. Z. Phys. **40**, 501 (1927)
- [19] W. Heisenberg, *Physikalische Prinzipien der Quantentheorie* (in German), (S. Hirzel Verlag, Leipzig, 1930).
- [20] P. Jordan, Naturwiss. **15**, 105 (1927).
- [21] P. Jordan, Naturwiss. **15**, 614 (1927); *ibid.*, 636 (1927).
- [22] P. Jordan, Z. Phys. **40**, 809 (1927).
- [23] M. Planck, Ann. Physik **4**, 553 (1901).

- [24] M. Planck, *Vorlesungen über die Theorie der Wärmestrahlung* (in German), 2. Aufl. (Verlag J. A. Barth, Leipzig, 1913).
- [25] Bohr, N.: On the Constitution of Atoms and Molecules. *Phil. Mag.* **26**,1 (1913)
- [26] N. Bohr, *Naturwiss.* **15**, 245 (1928).
- [27] N. Bohr, *Nature* **121**, 580 (1928).
- [28] W. Heisenberg, *Z. Phys.* **33**, 879 (1925).
- [29] M. Born, *Z. Phys.* **38**, 803 (1926).
- [30] M. Born, *Naturwiss.* **15**, 238 (1927).
- [31] E. Schrödinger, *Ann. Physik* **79**, 361 (1926).
- [32] E. Schrödinger, *Ann. Physik* **79**, 489 (1926).
- [33] E. Schrödinger, *Ann. Physik* **79**, 734 (1926).
- [34] E. Schrödinger, *Ann. Physik* **80**, 437 (1926).
- [35] E. Schrödinger, *Ann. Physik* **81**, 109 (1926).
- [36] E. Schrödinger, *Ann. Physik* **83**, 956 (1927).
- [37] W. Heisenberg, *Der Teil und das Ganze* (in German) 2nd ed. (dtv, München, 1975).
- [38] C. Eckart, *Phys. Rev.* **28**, 711 (1926).
- [39] M. Brieger, Research Report DLR-FB-2004-32 (ISSN 1434-8454), (German Aerospace Center, Cologne, 2004)
and its revised and extended online version
https://quantummist.eu/home/Schroedinger_I&II.pdf .
[https://www.researchgate.net/physics/search?=Michael Brieger](https://www.researchgate.net/physics/search?=Michael+Brieger)
- [40] M. Brieger, Research Report DLR-FB-2005-13 (ISSN 1434-8454), (German Aerospace Center, Cologne, 2005)
and its revised and extended online version
https://quantummist.eu/home/Schroedinger_III&IV.pdf
[https://www.researchgate.net/physics/search?=Michael Brieger](https://www.researchgate.net/physics/search?=Michael+Brieger)

- [41] P. A. M. Dirac, *The Principles of Quantum Mechanics*, 4th rev. ed. (Oxford University Press, Oxford, 1958 repr. 1982).
- [42] A. Messiah, *Quantum Mechanics*, Vol. I (North-Holland Publishing, Amsterdam, 1965).
- [43] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill, New York, 1981).
- [44] R. Courant and D. Hilbert, *Methoden der mathematischen Physik* (in German), 4th ed. (Springer, Berlin, 1993).
- [45] N. Bohr, Phil. Mag. **26**, 476 (1913).
- [46] N. Bohr, Phil. Mag. **26**, 857 (1913).
- [47] N. Bohr, Phil. Mag. **27**, 506 (1914).
- [48] N. Bohr, Phil. Mag. **30**, 394 (1915).
- [49] E. U. Condon and Halis Odabasi, *Atomic Structure*, (Cambridge University Press, Cambridge, 1980).
- [50] W. H. Zurek, Phys. Rev. D **24**, 1516 (1981).
- [51] W. H. Zurek, Phys. Rev. D **26**, 1862 (1982).
- [52] W. H. Zurek, Phys. Today **44**, 36 (Oct. 1991).
- [53] E. Schrödinger, Naturwiss. **23**/49, 823 (1935).
- [54] C. Cohen-Tannoudji, B. Diu, and F. Laloë, *Quantum Mechanics* Vol. I (J. Wiley & Sons, New York and Hermann, Paris, 1977).
- [55] H. Everett, Rev. Mod. Phys. **29**, 454 (1957).
- [56] F. M. Pipkin, *Atomic Physics Tests of the Basic Concepts in Quantum Mechanics* in: *Advances in Atomic and Molecular Physics* Vol. **14**, 281; D. R. Bates and B. Bederson eds. (Academic Press, New York, 1978).
- [57] N. Bohr, Z. Phys. **13**, 117 (1923).
- [58] L. de Broglie, Ann. de Physique **3**, 22 (1925).
- [59] L. de Broglie, Nature **112**, 540 (1923).
- [60] W. Paul, Rev. Mod. Phys. **62**, 531 (1990).